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ON *IS*-SPACES IN A FINSLER SPACE

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Introduction. In previous papers [4][1], [5], we have introduced a *TM*-connection $TM\Gamma$ on an n-dimensional Finsler space M from the standpoint of tangent Minkowski spaces and proved that M is a G-Landsberg space with respect to $TM\Gamma$, that is, its hv-curvature tensor $\widetilde{P}^i_{jkh}$ vanishes if and only if the *TM*-connection in consideration is the *IS*-connection. A G-Landsberg space with respect to the Cartan connection is nothing but an ordinary Landsberg space. A Finsler space M is called an *IS*-space if M admits the *IS*-connection $IS\Gamma$. Then an *IS*-space is always hv-flat ($\widetilde{P}^i_{jkh}=0$) and each indicatrix I_x(x : any point of M) as a Riemannian space is isometric under the parallel displacement with respect to $IS\Gamma$. Further the $IS\Gamma$ is also r-metrical, namely $Dg_{ij}=0$. As a special case, the $IS\Gamma$ involves the Berwald connection and the corresponding *IS*-space becomes a Landsberg space. When $n=2$, the above case alone occurs. In a previous paper [5], a condition was found for M to be a non-Landsberg *IS*-space. With respect to this space, however, there still many problems to be solved. For example, what special properties does it possess? Does such a space really exist? And so on.

In the present paper, we shall discuss the above problems and develop the theory of this space. Since the *IS*-connection is a *GT*-connection of *SK*-type, we investigate, in § 1, properties of the latter generally. In § 2, we find a special property of an *IS*-space and consider applications of it to other cases. As a consequence, we have that if a C-reducible Finsler space M is an *IS*-space then M is a Riemannian space or a Berwald space when $n\geqq 4$, and then M can be a *LCP*-space when $n=3$. In § 3 and § 4, we study a Riemannian *IS*-space and a C-reducible Berwald *IS*-space respectively in detail. In either of these cases, the h-connection of the *IS*-con-

1) Numbers in brackets refer to the references at the end of the paper.

nection depends on only the positional arguments. These spaces may be considered as special *IS*-spaces. The last section is devoted to the study of a *C*-reducible *LCP*-space. This space is an interesting example as an *IS*-space and has a noteworthy property, that is, it admits a *LCP*-frame.

The terminologies and notations refer to the papers ([4], [5], [6]) unless otherwise stated.

§1. *GT*-connections. Let M be an n-dimesional Finsler space with a fundamental function $L(x, y)$ and be endowed with a TM-connection $TM\Gamma = (\Gamma^{\ i}_{j\ k}, \Gamma^{\ i}_{k}, C^{i}_{j\ k})$. Then $\Gamma^{\ i}_{j\ k}$ and $\Gamma^{\ i}_{k}$ are given by

$$\Gamma^{\ i}_{j\ k} = G^{i}_{j\ k} + T^{i}_{j\ k} + Q^{i}_{j\ k}\ ,\quad \Gamma^{\ i}_{k} = y^{j}\,\Gamma^{\ i}_{j\ k}\ . \tag{1. 1}$$

Let M be *hv*-torsion-free with respect to $TM\Gamma$, namely

$$\Gamma^{\ i}_{i\ k} = G^{i}_{j\ k} + T^{i}_{j\ k} \quad (Q^{i}_{j\ k} = 0). \tag{1. 2}$$

We shall call a TM-connection defined by (1. 2) a *GT-connection.* The *h*-curvature tensor with respect to a TM-connection is given by

$$\begin{aligned} &R^{i}_{j\ kh} = \widetilde{K}^{i}_{j\ kh} + C^{i}_{j\ r}\ R^{r}_{kh}\ ,\quad R^{i}_{kh} = y^{j}\,R^{i}_{j\ kh} = y^{j}\,\widetilde{K}^{i}_{j\ kh}\ ,\\ &\widetilde{K}^{i}_{j\ kh} = \delta_{h}\,\Gamma^{\ i}_{j\ k} - \delta_{k}\,\Gamma^{\ i}_{j\ h} + \Gamma^{\ r}_{j\ k}\,\Gamma^{\ i}_{r\ h} - \Gamma^{\ r}_{j\ h}\,\Gamma^{\ i}_{r\ k}\ . \end{aligned} \tag{1. 3}$$

From (1. 2) and (1. 3) we have

$$\begin{aligned} &\widetilde{K}^{i}_{j\ kh} = H^{i}_{j\ kh} + \overset{(1)}{T}{}^{i}_{j\ kh} - \overset{(1)}{T}{}^{i}_{j\ hk}\ ,\quad \overset{(1)}{T}{}^{i}_{j\ kh} = \partial\, T^{i}_{j\ k}\,/\,\partial\, x^{h} - T^{r}_{h}\ G^{i}_{j\ kr}\\ &\qquad - \Gamma^{\ r}_{h}\ T^{i}_{j\ k\parallel r} + \Gamma^{\ r}_{j\ k}\ T^{i}_{rh} + T^{r}_{j\ k}\ G^{i}_{rh}\ , \end{aligned} \tag{1. 4}$$

where $H^{i}_{j\ kh}$ is the curvature tensor of Berwald.

On the other hand, the curvature tensor R^{i}_{kh} with respect to the non-linear connection $\Gamma^{\ i}_{k}$ is given by

$$\begin{aligned} &R^{i}_{kh} = \delta_{h}\,\Gamma^{\ i}_{k} - \delta_{k}\,\Gamma^{\ i}_{h} = y^{j}\,K^{i}_{j\ kh} + \overset{(1)}{T}{}^{i}_{kh} - \overset{(1)}{T}{}^{i}_{hk}\ ,\\ &\overset{(1)}{T}{}^{i}_{kh} = y^{j}\,\overset{(1)}{T}{}^{i}_{j\ kh} = \partial\, T^{i}_{k}\,/\,\partial x_{h} + \Gamma^{\ r}_{k}\ T^{i}_{rh} + T^{r}_{k}\ G^{i}_{rh}\ , \end{aligned} \tag{1. 5}$$

where K^{i}_{jkh} is the curvature tensor of Rund. Then it is verified that the following relations hold:

$$\begin{aligned} &\partial^{2}\, T^{i}_{k}\,/\,\partial\, y^{j}\ \partial\, x^{h} = \partial\, T^{i}_{j\ k}\,/\,\partial\, x^{h}\ ,\quad \Gamma^{\ i}_{k\parallel j} = G^{i}_{j\ k} + T^{i}_{j\ k}\ ,\\ &\Gamma^{\ r}_{k}\ T^{i}_{rh\parallel j} = \Gamma^{\ r}_{k}\ T^{i}_{j\ h\parallel r}\ ,\quad (y^{r}\ K^{i}_{rkh})_{\parallel j} = H^{i}_{j\ kh}\ . \end{aligned} \tag{1. 6}$$

From (1. 4) ~ (1. 6), we can state

Lemma 1. *With respect to a GT-connection, the following relation holds:*

(1. 7) $$\partial R^i_{kh} / \partial y^j = \widetilde{K}^i_{j\,kh} .$$

We shall say a Finsler space M to be *n-flat* and *h-flat* with respect to the connection in consideration if the curvature tensors R^i_{kh} and $R^i_{j\,kh}$ vanish respectively. It is seen from (1. 3) that if M is h-flat, then M is also n-flat. From (1. 3) and Lemma 1, we can state

Theorem 1. *With respect to a GT-connection, a Finsler space M is h-flat if and only if M is n-flat.*

Using $R_{j\,i\,kh} = g_{i\,r}\, R^r_{j\,kh}$ we put

(1. 8) $$K(x,\ y,\ X) = \frac{R_{j\,i\,kh}\, y^j\ y^k\ X^i\ X^h}{(g_{j\,k}\, g_{i\,h} - g_{j\,h}\, g_{i\,k}\,)\, y^j\ y^k\ X^i\ X^h} .$$

Then we shall call $K\,(x,\ y,\ X)$ the *sectional curvature* defined by y^i and X^i with respect to the connection in consideration. Especially we shall say M to be *of scalar curvature* $K\,(x,\ y)$ or to be *of constant curvature* K if $K\,(x,\ y,\ X)$ is independent of X^i or K is a constant respectively. From (1. 3) and (1. 5), we have

(1. 9) $$R_{o\,i\,o\,h} = K_{o\,i\,o\,h} - T_{i\,h|o} + T_{i\,r}\ T^r_h ,$$

where the index 0 means contraction by the vector y^i and the short vertical line indicates the covariant differentiation of Cartan.

Applying (1. 9) to (1. 8), we have

(1. 10) $$(K_{o\,i\,o\,h} - KL^2\, h_{i\,h} - T_{i\,h|o} + T_{i\,r}\ T^r_h\,)\, X^i\ X^h = 0 .$$

If (1. 10) holds for any X^i , then we obtain

(1. 11) $$2(K_{o\,i\,o\,h} - KL^2\, h_{i\,h}\,) - (T_{i\,h|o} + T_{h\,i\,|o}\,) + (T_{i\,r}\ T^r_h + T_{h\,r}\ T^r_i\,) = 0.$$

Conversely if (1. 11) holds, then (1. 8) holds for any X^i . Hence we have

Theorem 2. *With respect to a GT-connection, a Finsler space M is of scalar curvature K(x, y) if and only if an equation (1. 11) holds.*

Let M be n-flat with respect to a GT-connection. Then from (1. 5) we obtain

$$(1.\ 12)\quad \begin{aligned} R^i_{kh} &= K^i_{kh} + T^i_{k|h} - T^i_{h|k} + T^r_k T^i_{rh} - T^r_h T^i_{rk} \\ &+ T^r_k P^i_{rh} - T^r_h P^i_{rk} = 0, \text{ where } K^i_{kh} = y^j K^i_{jkh}. \end{aligned}$$

Contracting (1. 12) by y^k, we have

$$(1.\ 13)\quad K^i_{oh} - T^i_{h|o} + T^i_r T^r_h = 0,$$

which implies

$$(1.\ 14)\quad K_{ioh} - T_{ih|o} + T_{ir} T^r_h = 0.$$

Let this connection be of *SK*-type, namely $T_{ih} + T_{hi} = 0$. Then it follows from this property that $T_{ir} T^r_h = T_{hr} T^r_i$. On the other hand, the tensor K_{ioh} is also symmetric in i and h. Therefore in view of (1. 14), we have

$$(1.\ 15)\quad K_{ioh} + T_{ir} T^r_h = 0,\quad T_{ih|o} = 0.$$

From (1. 13) and (1. 15) we have $K^i_{oh} = -T^i_r T^r_h$, differentiation of which by y^k yields

$$(1.\ 16)\quad K^i_{oh\|k} = -T^i_{kr} T^r_h - T^i_r T^r_{kh}.$$

It is known [3] that the following identity holds:

$$(1.\ 17)\quad K^i_{kh} = \tfrac{1}{3}(K^i_{oh\|k} - K^i_{ok\|h}).$$

Substituting (1. 16) in (1. 17), we have

$$(1.\ 18)\quad K^i_{kh} = \tfrac{1}{3}(T^r_{hk} - T^r_{kh})T^i_r + \tfrac{1}{3}(T^i_{hr} T^r_k - T^i_{kr} T^r_h),$$

substitution of which in (1. 12) yields

$$(1.\ 19)\quad \begin{aligned} & T^i_{k|h} - T^i_{h|k} + \tfrac{1}{3}(T^r_{hk} - T^r_{kh})T^i_r + (T^i_{rh} + \tfrac{1}{3}T^i_{hr} + P^i_{rh})T^r_k \\ & - (T^i_{rk} + \tfrac{1}{3}T^i_{kr} + P^i_{kr})T^r_h = 0. \end{aligned}$$

Conversely if (1. 18) and (1. 19) hold, then it is easily verified that equations (1. 12) and (1. 15) hold.

Hence we can state

Theorem 3. *With respect to a GT-connection of SK-type, M is of n-flat (or h-flat) if and only if equation (1. 18) and (1. 19) hold.*

Under the same condition, the equation (1. 11) is reducible to

(1. 20) $\quad K^i_{oh} = KL^2 h^i_h - T^i_r \; T^r_h$

Then we can state

Theorem 4. *With respect to a GT-connection of SK-type, M is of scalar curvature K if and only if an equation (1. 20) holbs.*

We shall seek for another condition. Since $K^i_{oh\|k} = K^i_{oh}|_k + C^r_{hk} \; K^i_{or} - C^i_{rk} \; K^r_{oh}$, we differentiate (1. 20) by y^k v-covariantly and apply the result to (1. 17). Further noticing that $T^i_r|_k = T^i_{kr} + C^i_{sk} \; T^s_r - C^s_{rk} \; T^i_s$, we obtain

(1. 21)
$$3K^i_{kh} = L^2 (K|_k \; h^i_h - K|_h \; h^i_k) + 3K(y_k \; h^i_h - y_h \; h^i_k)$$
$$+ T^i_{hr} \; T^r_k - T^i_{kr} \; T^r_h + (T^r_{hk} - T^r_{kh}) T^i_r \; .$$

Conversely, contracting (1. 21) by y^k, we can obtain (1. 20).
Consequently we can state

Theorem 5. *With respect to a GT-connection of SK-type, M is of scalar curvature K if and only if an equation (1. 21) holds.*

Let the scalar K be a constant. Then the equation (1. 21) is reducible to

(1. 22) $\quad K^i_{kh} = K(y_k \; h^i_h - y_h \; h^i_k) + \widetilde{T}^i_{kh}$,

where $\widetilde{T}^i_{kh} = \{T^i_{hr} \; T^r_k - T^i_{kr} \; T^r_h + (T^r_{hk} - T^r_{kh}) T^i_r\} \; / 3.$

Differentiating (1. 22) by y^j, we have

(1. 23) $\quad H^i_{jkh} = K(g_{jk} \; \delta^i_h - g_{jh} \; \delta^i_k) + \widetilde{T}^i_{jkh}$,

where $\widetilde{T}^i_{jkh} = \widetilde{T}^i_{kh\|j}$.

Suppose that the equation (1. 23) holds. Then by contracting (1. 23) by y^j we have (1. 22). Further if we differentiate (1. 22) by y^j and take account of (1. 23), then we have

$$K_{\|j} (y_k \; h^i_h - y^h \; h^i_k) = 0,$$

contraction of which with respect to i and h yields $(n-2) \; y_k K_{\|j} = 0$ and hence $K_{\|j} = 0$. On the other hand, we have a Bianchi's identity

(1. 24) $\quad K^i_{kh|j} + K^i_{jk|h} + K^i_{hj|k} + P^i_{kr} \; K^r_{hj} + P^i_{jr} \; K^r_{kh} + P^i_{hr} \; K^r_{jk} = 0.$

If we apply (1. 22) to (1. 24), then we obtain

$$(y_k \; h^i_h - y_h \; h^i_k) K_{|j} + (y_j \; h^i_k - y_k \; h^i_j) K_{|h} + (y_h \; h^i_j - y_j \; h^i_h) K_{|k}$$

$$+\widetilde{T}^i_{kh|j}+\widetilde{T}^i_{jk|h}+\widetilde{T}^i_{hj|k}+P^i_{kr}\widetilde{T}^r_{hj}+P^i_{jr}\widetilde{T}^r_{kh}+P^i_{hr}\widetilde{T}^r_{jk}=0,$$

contraction of which with respect to i and h yields

(1. 25) $$(n-2)(y_k K_{|j}-y_j K_{|k})+\widetilde{T}^r_{jk|r}+\widetilde{T}^r{}_{j|k}-\widetilde{T}^r_{rk|j}+P^s_{kr}\widetilde{T}^r_{sj}-P^s_{jr}\widetilde{T}^r_{sk}+P_r\widetilde{T}^r_{jk}=0.$$

If K is a constant, then from (1. 25) we obtain

(1. 26) $$\widetilde{T}^r_{jk|r}+\widetilde{T}^r_{rj|k}-\widetilde{T}^r_{rk|j}+P^s_{kr}\widetilde{T}^r_{sj}-P^s_{jr}\widetilde{T}^r_{sk}+P_r\widetilde{T}^r_{jk}=0.$$

Conversely if (1. 26) holds, then from (1. 25) we have

(1. 27) $$y_k K_{|j}-y_j K_{|k}=0.$$

Since $K_{\|j}=0$, the vector $K_{|j}$ is independent of y^i. Therefore if we differentiate (1. 27) by y^i, then we have $g_{ik}K_{|j}-g_{ij}K_{|k}=0$, contraction of which by g^{ik} yields $(n-1)K_{|j}=0$. Thus we can state

Theorem 6. *With respect to a GT-connection of SK-type, M is of constant curvature K if and only if equations (1. 23) and (1. 26) hold.*

§ 2. *IS*-spaces. Let M be an *IC*-space. Then there exists an indicatric tensor T^i_j such that

(2. 1) $$T_{ij}+T_{ji}=0, \text{ where } T_{ij}=g_{ir}T^r_j,$$

(2. 2) $$T_{ijk}+T_{jik}+2(C_{ijr}T^r_k+P_{ijk})=0, \text{ where } T_{ijk}=g_{jr}T^r_{ik}.$$

In this case, the following relations hold [6]:

(2. 3) $$T_{ri}C^r_{jk}+T_{rj}C^r_{ki}+T_{rk}C^r_{ij}=3P_{ijk},$$

(2. 4) $$T^i_{jh}+T^i_{hj}+P^i_{jh}+T^r_h C^i_{rj}+T^r_j C^i_{rh}-T^i_r C^r_{jh}=0,$$

(2. 5) $$T_{kjh}+T_{khj}+2T^r_j C_{rjk}+2T^r_h C_{rjk}=0,$$

(2. 6) $$T^i_{jk}+T^i_{kj}+4P^i_{jk}=0,$$

(2. 7) $$G^i_{jkh}+T^i_{jk\|h}-C^i_{jh;k}=0, \text{ where}$$

(2. 8) $$C^i_{jh;k}=C^i_{jh|k}-T^r_k C^i_{jh\|r}+P^i_{rk}C^r_{jh}-P_{rk}C^i_{rh}-P^r_{hk}C^i_{jr}+T^i_{rk}C^r_{jh}-T^r_{jk}C^i_{rh}-T^r_{hk}C^i_{jr}.$$

Differentiating (2. 6) by y^h , we have

(2. 9) $$T^i_{jk\|h} + T^i_{kj\|h} + 4P^i_{jk\|h} = 0.$$

By interchanging indices j and h in (2. 9), we have a similar expression. Then if we subtract the latter from (2. 9) and apply (2. 7) to the result, then we have

$$C^i_{kh;j} - C^i_{kj;h} + 4(P^i_{kj\|h} - P^i_{kh\|j}) = 0,$$

which implies

(2. 10) $$g_{is}(C^s_{kh;j} - C^s_{kj;h}) + 4(P_{kij\|h} - 2P^s_{kj}C_{ish} - P_{kih\|j} + 2P^s_{kh}C_{isj}) = 0.$$

By means of (2. 3), we have

(2. 11) $$3(P_{ikh\|j} - P_{ikj\|h}) = T^r_h C_{kri\|j} - T^r_j C_{kri\|h} + (T^r_{jh} - T^r_{hj})C_{kri} + T^r_{ji}C_{hrk} - T^r_{hi}C_{jrk} + T^r_{jk}C_{irh} - T^r_{hk}C_{irj}.$$

On the other hand, we obtain

(2. 12) $$g_{is}(C^s_{kj\|r}T^r_h - C^s_{kh\|r}T^r_j) = C_{kij\|r}T^r_h - C_{kih\|r}T^r_j + 2(C^s_{kh}C_{isr}T^r_j - C^s_{kj}C_{isr}T^r_h).$$

From a Bianchi's identity we have

(2. 13) $$C_{kih|j} - C_{kij|h} = P_{jkhi} - P_{hkji} + P^r_{ji}C_{khr} - P^r_{hi}C_{kjr}.$$

By virtue of (2. 8) and (2. 10) ~(2. 13), we obtain

(2. 14) $$\begin{aligned} &2C^r_{kh}C_{ris}T^s_j - 2C^r_{jk}C_{ris}T^s_h + 2(C^r_{hk}P_{rij} - C^r_{jk}P_{rih}) \\ &+ 8(C^r_{ij}P_{rkh} - C^r_{ih}P_{rkj}) + (T^s_{hk} + T^s_{kh})C_{sij} \\ &- (T^r_{jk} + T^r_{kj})C_{rih} + (T_{hsi} - T_{sih})C^s_{jk} + (T_{sij} - T_{jsi})C^s_{kh} = 0. \end{aligned}$$

Applying (2. 2) and (2. 6) to (2. 14), we have

(2. 15) $$u_{ikhj} = u_{kihj}, \quad u_{ikhj} = C^r_{ij}P_{rkh} - C^r_{ih}P_{rkj}.$$

In this case, the following relation holds:

(2. 16) $$u_{ikjh} = -u_{ikhj}.$$

We shall say a Finsler space M to be *pseudo-CP-symmetric* if M satisfies (2. 15). Applying (2. 3) to (2. 15), we have

(2. 17) $$T_{ri}\, S^r_{kjh} - T_{rk}\, S^r_{ijh} + T_{rj}\, S^r_{hik} - T_{rh}\, S^r_{jik} = 0,$$

contraction of which by g^{ij} yields

(2. 18) $$T^r_k\, S_{rh} + T^r_h\, S_{rk} = T^{rs}\, (S_{krsh} + S_{hrsk}).$$

Since tensors T^{rs} and $(S_{krsh} + S_{hrsk})$ are skew-symmetric and symmetric in indices r and s respectively, from (2. 18) we have

(2. 19) $$S_{rk}\, T^r_h + S_{rh}\, T^r_k = 0.$$

Consequently we can state

Theorem 7. *An IS-space is pseudo-CP-symmetric. In this case, relations (2. 17) and (2. 19) hold.*

A Finsler space M will be said to be *CP-related* if M satisfies

(2. 20) $$P_{ijk} = \mu\,(x, y) C_{ijk},$$

where $\mu\,(x, y)$ is a positively homogeneous scalar of degree 1 in y^i.

In this case if we substitute (2. 20) in (2. 15), then we have $\mu\,(x, y) = 0$ or $S_{ikhj} = 0$. Hence we can state

Theorem 8. *If an IS-space is CP-related, then M is a Landsberg space or v-flat.*

Let M be *P-reducible*, that is, the tensor P_{ijk} is expressible in

(2. 21) $$P_{ijk} = (P_i\, h_{jk} + P_j\, h_{ik} + P_k\, h_{ij})\,/(n+1) \quad (n \geqq 3),$$

where $P_i = P_{ijk}\, g^{jk}$. Then applying (2. 21) to (2. 15), we obtain

(2. 22) $$C^r_{ij}\, P_r\, h_{kh} - C^r_{ih}\, P_r\, h_{kj} = C^r_{kj}\, P_r\, h_{ih} - C^r_{kh}\, P_r\, h_{ij},$$

conraction of which by g^{kh} yields

(2. 23) $$(n-3) C_{ijr}\, P^r + P_r\, C^r\, h_{ij} = 0, \quad \textit{where}\ C^r = C^r_{kh}\, g^{kh}.$$

Further if we contract (2. 23) by g^{ij}, then we have

(2. 24) $$2(n-2) P_r\, C^r = 0.$$

Consequently we can state

Theorem 9. *If an IS-space M is P-reducible, then a relation (2. 22) holds and the vectors* C_i *and* P_i *are mutually orthogonal, i.e.* , $P_r C^r = 0$. *When* $n \geqq 4$, *the* torsion tensor C_{ijk} *is orthogonal to the vector* P^i , *i.e.* , $C_{ijk} P_i = 0$.

We put

$$C = (C_i C^i)^{\frac{1}{2}},\ C^2 = C_i C^i,\ P = (P_i P^i)^{\frac{1}{2}},\ P^2 = P_i P^i, \tag{2. 25}$$
$$m^i = C^i / C,\ n^i = P^i / P,\ m_i = C_i / C,\ n_i = P_i / P.$$

Especially when $n=3$, we shall call a frame (l^i, m^i, n^i) an *LCP-frame* if the frame is orthonormal. Further a three dimensional Finsler space M will be called a *LCP-space* if M admits an *LCP*-frame.

Then we can state.

Corollary 9. 1. *A P-reducible IS-space of dimension 3 can be an LCP-space.*

Let M be C-reducible, that is, the tensor C_{ijk} is expressible in

$$C_{ijk} = (C_i h_{jk} + C_j h_{ik} + C_k h_{ij}) / (n+1) \quad (n \geqq 3). \tag{2. 26}$$

Then it is easily seen from (2. 26) that M is also P-reducible. Therefore substituting (2. 26) in (2. 22) and making use of (2. 24), we have

$$(C_i P_j + P_i C_j) h_{kh} + (C_k P_h + P_k C_h) h_{ij} = (C_i P_h + P_i C_h) h_{jk} + (C_j P_k + P_j C_k) h_{ih}, \tag{2. 27}$$

contraction of which by g^{kh} yields

$$(n-3)(C_i P_j + P_i C_j) = 0. \tag{2. 28}$$

When $n \geqq 4$, from (2. 24) and (2. 28) we have $C_i = 0$ or $P_i = 0$. Because of (2. 26), the former implies that M is a Riemannian space. It follows from the latter and (2. 21) that M is a Landsberg space. On the other hand, it is known [2] that a C-reducible Landsberg space is reduced to a Berwald space. Therefore M is a Berwald space. Hence we can state

Lemma 2. *If an n-dimensional IS-space* $M (n \geqq 4)$ *is C-reducible, then M is a Riemannian space or a Berwald space.*

Let n be equal to 3. Then contracting (2. 27) by $C^k P^h$, we have

$$C^2 P^2 h_{ij} = P^2 C_i C_j + C^2 P_i P_j \tag{2. 29}$$

If $C^2 P^2 \neq 0$, then from (2. 25) and (2. 29) we obtain

$$h_{ij} = m_i m_j + n_i n_j , \tag{2. 30}$$

which indicates that a frame (l^i, m^i, n^i) is orthonomal without fail. Consequently we can state

Lemma 3. *If $C^2 P^2 \neq 0$, then a three-dimensional C-reducible IS-sqace is an LCP -space.*

§ 3. Riemannian *IS*-spaces. Let M be a Riemannian *IS*-space, that is, M is a Riemannian space which admits the *IS*-connection. Then there exist tensors T^i_k and T^i_{jk} such that the following relations hold for a vector y^i :

$$T^i_{jk\|h} = 0,\ T^i_{jk} + T^i_{kj} = 0,\ T_{jik} + T_{ijk} = 0,\ T_{ijk} + T_{ikj} = 0, \tag{3. 1}$$

$$T^i_k = y^j T^i_{jk},\ T^i_k y_i = T^i_k y^k = 0,\ T_{ij} + T_{ji} = 0. \tag{3. 2}$$

In this case, the *IS*-connection $(\Gamma^i_{jk}, \Gamma^i_k, 0)$ is given as follows:

$$\Gamma^i_{jk} = \{{}_j{}^i{}_k\}(x) + T^i_{jk}(x),\ \Gamma^i_k = y^j \Gamma^i_{jk} , \tag{3. 3}$$

where $\{{}_j{}^i{}_k\}$ are the Christoffel symbols formed with $g_{ij}(x)$.

If the *IS*-connection is symmetric, then from (3. 1) and (3. 3) we have $T^i_{jk} = 0$. Hence we can state

Lemma 4. *The IS-connection of a Riemannian space is symmetric if and only if the h-connection Γ^i_{jk} is the Riemannian connection $\{{}_j{}^i{}_k\}$.*

Let M be n-flat. Then by virtue of (1. 18), (1. 19) and (3. 1) we have

$$K^i_{kh} = (2T^r_{hk} T^i_r + T^i_{hr} T^r_k - T^i_{kr} T^r_h) / 3, \tag{3. 4}$$

$$T^i_{k|h} - T^i_{h|k} + \tfrac{2}{3}(T^i_{rh} T^r_k - T^i_{rk} T^r_h + T^r_{hk} T^i_r) = 0. \tag{3. 5}$$

In this case we have $T^i_{k|s} y^s = 0$, differentiation of which by y^h yields because of (3. 1) and (3. 2)

$$T^i_{k|h} = y^s \nabla_s T^i_{kh} ,\ (T^i_{k|h})_{\|j} = \nabla_j T^i_{kh} , \tag{3. 6}$$

where the symbol ∇ indicates the covariant differentiation with respect to $\{{}_j{}^i{}_k\}$. We denote the Riemannian curvature tensor by $\overset{r}{R}{}^i_{jkh}$. Differentiating (3. 4) and (3. 5) by y^j and making use of (3. 1) and (3. 6) we obtain

(3. 7) $\overset{r}{R}{}^{i}_{jkh} = (2T^{r}_{hk} T^{i}_{jr} + T^{i}_{jr} T^{r}_{jk} - T^{i}_{kr} T^{r}_{jh})/3,$

(3. 8) $\nabla_j T^{i}_{kh} = (T^{i}_{rk} T^{r}_{jh} - T^{i}_{rh} T^{r}_{jk} + T^{r}_{kh} T^{i}_{jr})/3.$

Consequently we can state

Theorem 10. *A Riemannian IS-space M is h-flat (or n-flat) with respect to the IS-connection if and only if equations (3. 7) and (3. 8) hold.*

Let M be of scalar curvature $K(x, y)$. Then from (1. 20) we have

$$K_{oioh} - KL^2 h_{ih} + T_{ir} T^{r}_{h} = 0,$$

which is expressible in

(3. 9) $\{\overset{r}{R}_{jikh} - K(g_{jk} g_{ih} - g_{jh} g_{ik}) + T_{jir} T^{r}_{kh}\} y^j y^k = 0.$

If (3. 9) holds for any vector y^i, then we obtain

(3. 10) $$\overset{r}{R}_{jikh} + \overset{r}{R}_{kijh} - K(2g_{jk} g_{ih} - g_{jh} g_{ik} - g_{kh} g_{ij}) + T_{jir} T^{r}_{kh} + T_{kir} T^{r}_{jh} = 0.$$

By interchanging indices k and h in (3. 10) we obtain a similar equation. Subtracting the latter from (3. 10) and using $\widetilde{T}_{jikh}$ in (1. 23) we have

(3. 11) $\overset{r}{R}_{jikh} = K(g_{jk} g_{ih} - g_{jh} g_{ik}) + \widetilde{T}_{jikh},$

where $\widetilde{T}_{jikh} = (T_{hir} T^{r}_{jk} - T_{kir} T^{r}_{jh} + 2T^{r}_{hk} T_{jir})/3.$

In this case, the condition (1. 26) is, because of (3. 6), reducible to

$$y^s(\nabla_r T^{r}_{kt} \cdot T^{t}_{sj} - \nabla_r T^{r}_{jt} \cdot T^{t}_{sk} + 2\nabla_r T^{t}_{kj} \cdot T^{r}_{st} + 3\nabla_k T^{r}_{jt} \cdot T^{t}_{sr} - 3\nabla_j T^{r}_{kt} \cdot T^{t}_{sr} + 4\nabla_s T^{r}_{jt} \cdot T^{t}_{kr} - 4\nabla_s T^{r}_{kt} \cdot T^{t}_{jr})/3 = 0,$$

differentiation of which by y^h yields

(3. 12) $$\nabla_r T^{r}_{kt} \cdot T^{t}_{hj} - \nabla_r T^{r}_{jt} \cdot T^{t}_{hk} + 2\nabla_r T^{t}_{kj} \cdot T^{r}_{ht} + 3\nabla_k T^{r}_{jt} \cdot T^{t}_{hr} - 3\nabla_j T^{r}_{kt} \cdot T^{t}_{hr} + 4\nabla_h T^{r}_{jt} \cdot T^{t}_{kr} - 4\nabla_h T^{r}_{kt} \cdot T^{t}_{jr} = 0.$$

Thus we can state

Theorem 11. *A Riemannian IS-space M is of constant curvature K with respect to the IS-connection if and only if equations (3. 11) and (3. 12) hold.*

§ 4. *CRBIS*-spaces. In this and next sections, we shall consider non-Riemannian

C-reducible *IS*-spaces. If we apply (2. 21) and (2. 26) to (2. 3), then we have

$$C_r\,(T^r_i\,h_{jk}+T^r_j\,h_{ik}+T^r_k\,h_{ij})=3(P_i\,h_{jk}+P_j\,h_{ik}+P_k\,h_{ij}),$$

contraction of which by g^{jk} yields

(4. 1) $\quad C_r\,T^r_i=3P_i\,.$

If $n\geqq 4$, then M is, because of Lemma 2. a Berwald space, namely

(4. 2) $\quad P_i=0,\ P^i_{jk}=0,\ G^i_{jkh}=0,\ C^i_{jh|k}=0.$

Also when $n=3$, such a case may be considered.

From (2. 2) ~(2. 7), (4. 1) and (4. 2) we have

(4. 3) $\quad C_r\,T^r_i=0,\ C_r\,T^r_{ji}+C_{r\|j}\,T^r_i=0,$

(4. 4) $\quad T_{ijk}+T_{jik}+2(C_i\,T_{jk}+C_jT_{ik})/(n+1)=0,$

(4. 5) $\quad T_{kji}+T_{kij}+2(C_i\,T_{kj}+C_j\,T_{ki})/(n+1)=0,$

(4. 6) $\quad T^i_{jk}+T^i_{kj}=0,\ T_{jik}+T_{kij}=0,$

(4. 7) $\quad T^i_{jk\|h}=-T^r_k\,C^i_{jh\|r}+T^i_{rk}\,C^r_{jh}-T^r_{jk}\,C^i_{rh}-T^r_{hk}\,C^i_{jr}\,.$

Since $C_{ijk\|h}=C_{ijh\|k}$, we apply (2. 26) to this expression and contract the result by g^{jk}. Then we obtain

(4. 8) $\quad C_{i\|h}=ah_{ih}+2C_i\,C_h/(n+1)-(C_i\,l_h+l_i\,C_h)/L,$

where $a=C_{j\|k}\,g^{jk}/(n-1)-2C^2/(n^2-1)$. By the use of (4. 8) we have

(4. 9) $\quad C^i_{\|h}=bh^i_h-2C^i\,C_h/(n+1)-(C^i\,l_h+l^i\,C_h)/L,$

where $b=a-2C^2/(n+1)$. From (4. 3), (4. 4) and (4. 8) we have

(4. 10) $\quad C_r\,T^r_{ij}=C^r\,T_{irj}=-aT_{ij}\,,\ T^i_{jk}\,C^j=bT^i_k\,.$

If we substitute (2. 26) in (4. 7) and apply (4. 3), (4. 4) and (4. 8) ~ (4. 10) to the result, then we can obtain $T^i_{jk\|h}=0$. Thus we can state

Lemma 5. *If an IS-space is a C-reducible Berwald space, then the tensor* T^i_{jk} *is independent of* y^i.

If we differentiate the second equation in (4. 10) by y^h , then because of Lemma 5 we have

(4. 11) $T^i_{jk} C^j{}_{\|h} = b_{\|h} T^i_k + bT^i_{hk}$.

Further substituting (4. 9) in (4. 11) and using (4. 10), we obtain

$$T^i_k \{b_{\|h} + 2bl_h / L + 2bC_h / (n+1) + C_h / L^2\} = 0,$$

which, if $T^i_k \neq 0$, implies

(4. 12) $b_{\|h} + 2bl_h / L + 2bC_h / (n+1) + C_h / L^2 = 0.$

If we regard (4. 9) as a differential equation with respect to C^i , then it follows from (4. 9) that (4. 12) is a condition for (4. 9) to be integrable.

Similarly from (4. 8) (or from the first equation (4. 10)) we have

(4. 13) $a_{\|h} + 2al_h L - 2aC_h / (n+1) + C_h / L^2 = 0.$

Since $G^i_{jk\|h} = 0$ and $T^i_{jk\|h} = 0$, we have

(4. 14) $T^i_{k|h} = T^i_{jk|h} y^j$, $(T^i_{k|h})_{\|j} = T^i_{jk|h}$.

If we put

(4. 15) $\overline{T}^i_{kh} = T^i_{k|h} - T^i_{h|k} + T^r_k T^i_{rh} - T^r_h T^i_{rk}$,

then from (4. 14) we have

(4. 16) $\overline{T}^i_{jkh} = \overline{T}^i_{kh\|j} = T^i_{jk|h} - T^i_{jh|k} + T^r_{jk} T^i_{rh} - T^r_{jh} T^i_{rk}$.

In this case, from (1. 6) and (1. 12) we obtain

(4. 17) $R^i_{kh} = K^i_{kh} + \overline{T}^i_{kh}$, $\widetilde{K}^i_{jkh} = H^i_{jkh} + \overline{T}^i_{jkh}$.

We shall call a Finsler space M a *CRBIS-space* if M is a C-reducible Berwald space and an *IS*-space. If $R^i_{kh} = \overline{T}^i_{kh}$ or $\widetilde{K}^i_{jkh} = \overline{T}^i_{jkh}$, then it follows from (4. 17) that this space is a locally Minkowski space.

Thus we can state

Theorem 12. *In a CRBIS-space, the h-connection* Γ^i_{jk} *of the IS-connection is independent of* y^i . *The scalars a and b in (4. 8) and (4. 9) satisfy (4. 10), (4. 12) and*

(4. 13). The tensors R^i_{kh} *and* $\widetilde{K}^i_{j\,kh}$ *are given by (4. 17). Especially if* $R^i_{kh} = \overline{T}^i_{kh}$ *or* $\widetilde{K}^i_{j\,kh}=\overline{T}^i_{j\,kh}$, *then the space in consideration becomes a locally Minkowski space.*

Let M be n-flat. Then from (1. 15) and (4. 14) we have

$$T^i_{k\,|\,h} = T^i_{kh\,|\,s}\,y^s\ ,\ (T^i_{k\,|\,h})_{\,\|\,j} = T^i_{kh\,|\,j}\ , \tag{4. 18}$$

which corresponds to (3. 6) in §3. Therefore in the same way as in §3, we odtain

$$H^i_{j\,kh} = (2T^r_{hk}\,T^i_{j\,r} + T^i_{hr}\,T^r_{j\,k} - T^i_{kr}\,T^r_{j\,h})/3, \tag{4. 19}$$

$$T^i_{kh\,|\,j} = (T^i_{rk}\,T^r_{j\,h} - T^i_{rh}\,T^r_{j\,k} + T^r_{kk}\,T^i_{j\,r})/3. \tag{4. 20}$$

Hence we can state

Theorem 13. *A CRBIS-space is h-flat (or n-flat) with respect to the IS-connection if and only if equations (4. 19) and (4. 20) hold.*

From (4. 19) and (4. 20) we have $H^i_{j\,kh} = T^i_{kh\,|\,j} + T^r_{kh}\,T^i_{rj}$. Therefore if the following relation holds, then the tensor H^i_{jkh} vanishes, that is, M is a locally Minkowski space:

$$T^i_{kh\,|\,j} + T^r_{kh}\,T^i_{rj} = 0. \tag{4. 21}$$

Conversely let M be a locally Minkowski space. Then we have $K^i_{kh}=0$ and $H^i_{j\,kh}=0$. Therefore it follows from (4. 17) and (4. 21) that the tensors R^i_{kh} and $\widetilde{K}^i_{j\,kh}$ both vanish. Consequently we can state

Corollary 13. 1. *If we can choose a tensor* $T^i_{j\,k}$ *such that an equation (4. 21) holds, then a CRBIS-space is a locally Minkowski space if and only if M is h-flat (or n-flat) with respect to the IS-connection.*

Note 1. Corresponding to this corollary, a similar corollary can be obtained for Theorem 10.

The tensor $H_{j\,i\,kh}$ in this section and the tensor $\overset{r}{R}_{j\,i\,kh}$ in §3 have the similar property. Therefore corresponding to (3. 11) and (3. 12), we obtain

$$H_{j\,i\,kh} = K(g_{j\,k}\,g_{i\,h} - g_{j\,h}\,g_{i\,k}) + \widetilde{T}_{j\,i\,kh}\ , \tag{4. 22}$$

where $\widetilde{T}_{j\,i\,kh} = (T_{hi\,r}\,T^r_{j\,k} - T_{ki\,r}\,T^r_{j\,h} + 2T^r_{hk}\,T_{j\,i\,r})/3$.

$$T^r_{kt\,|\,r}\,T^t_{hj} - T^r_{jt\,|\,r}\,T^t_{hk} + 2T^t_{kj\,|\,r}\,T^r_{ht} + 3T^r_{jt\,|\,k}\,T^t_{hr} - 3T^r_{kt\,|\,j}\,T^t_{hr} + 4T^r_{jt\,|\,h}\,T^t_{kr} - 4T^r_{kt\,|\,h}\,T^t_{j\,r} = 0. \tag{4. 23}$$

Hence we can states

Theorem 14. *With respect to the IS-connection, a CRBIS-space M is of constant curvature K in the sense of Riemannian geometry if and only if equations (4. 22) and (4. 23) hold.*

§ 5. C-reducible *LCP*-spaces. Throughout this section, we assume $c^2 p^2 \neq 0$. Let M be a three-dimensional C-reducible IS-space. Then because of Lemma 3, M is an LCP-space. We put

(5. 1) $$T^i_k = f(x, y)L(x, y)a^i_k, \quad a^i_k = m^i n_k - n^i m_k,$$

where $f(x, y)$ is a positively homogeneous scalar of degree 0 in y^i. From (2. 25), (4. 1) and (5. 1) we have

(5. 2) $$f(x, y)L(x, y)C = 3P.$$

Applying (2. 21) to (2. 6), we have

(5. 3) $$T^i_{jk} + T^i_{kj} + P(n^i h_{jk} + n_j h^i_k + n_k h^i_j) = 0.$$

By the use of (2. 26) and (2. 30), we obtain

(5. 4) $$m^i{}_{\|j} = m^i|_j - C(3m^i m_j + n^i n_j)/4,$$
$$m^i{}_{\|j} = m_i|_j + C(3m_i m_j + n_i n_j)/4,$$

(5. 5) $$n^i{}_{\|j} = n^i|_j - C(m^i n_j + n^i m_j)/4,$$
$$n_{i\|j} = n_i|_j + C(m_i n_j + n_i m_j)/4.$$

If we differentiate (5. 1) by y^j and make use of (5. 1), (5. 2), (5. 4) and (5. 5), then we have

(5. 6) $$T^i_{jk} = (Lf_{\|j} + fl_j)a^i_k + fL(m^i|_j n_k - n^i|_j m_k + m^i n_k|_j - n^i m_k|_j)$$
$$-3P(m^i a_{jk} + n^i h_{jk})/2, \quad a_{jk} = g_{jr} a^r_k.$$

If we put $d_k = Ln^j|_k m_j$, then we have

(5. 7) $$Lm_j|_k n^j = Lm^j|_k n_j = -d_k.$$

Substituting (5. 6) in (5. 3), we obtain

(5. 8) $$L(f_{\|j} a^i_k + f_{\|k} a^i_j) + f(l_j a^i_k + l_k a^i_j) + P(n_j h^i_k + n_k h^i_j - 2n^i h_{jk})$$
$$+ fL\{m^i|_j n_k + m^i|_k n_j - n^i|_j m_k - n^i|_k m_j + m^i(n_k|_j + n_j|_k)$$

$$-n^i (m_k|_j + m_j|_k)\} = 0.$$

It is easily seen that following relations hold:

$$(5.\ 9)\quad \begin{aligned} &a^i_k m^k = -n^i,\ a^i_k n^k = m^i,\ a^i_k m_i = n_k,\ a^i_k n_i = -m_k,\\ &f_{\|j} l^j = 0,\ m^i|_j m_i = m_i|_j m^i = n^i|_j n_i = n_i|_j n^i = 0. \end{aligned}$$

If we contract (5. 8) by $m^j m^k n_i$ and $n^j n^k m_i$ respectively, then on making use of (5. 9) we have

$$(5.\ 10)\qquad Lf_{\|j} m^j = -P,\quad f_{\|j} n^j = 0.$$

Therefore the vector $Lf_{\|j}$ is expressible in the form

$$(5.\ 11)\qquad Lf_{\|j} = -Pm_j\ .$$

Since $Lm^i|_j l_i = -m_j$, because of (5. 7) and (5. 9) we obtain

$$(5.\ 12)\qquad Lm^i|_j = -l^i m_j - n^i d_j\ .$$

Similarly for $Ln^i|_j$, we have

$$(5.\ 13)\qquad Ln^i|_j = -l^i n_j + m^i d_j\ .$$

Conversely if (5. 11) ~(5. 13) hold, then it is verified that (5. 8) holds.

Substituting (5. 11) ~(5. 13) in (5. 6), we have

$$(5.\ 14)\qquad T^i_{jk} = f(l_j a^i_k - l_k a^i_j - l^i a_{jk}) - P(2m_j a^i_k + 3m^i a_{jk} + 3n^i h_{jk})/2.$$

In this case, we can prove by the use of (5. 2) and C-reducibility that tensors T^i_k and T^i_{jk} given by (5. 1) and (5. 14) satisfy (2. 1) and (2. 2), and further from (5. 11)~(5. 13) that $T^i_{k\|j} = T^i_{jk}$. Thus we can state

Lemma 6. *If a C-reducible LCP-space M satisfies (5. 2) and (5. 11)~(5. 13), then M is an IS-space.*

Since $C^2 = g_{ij} C^i C^j$ and $C^i = Cm^i$, from (2. 26) and (4. 8) we have

$$(5.\ 15)_1\quad \begin{aligned} &m^i_{\|j} = b(h^i_j - m^i m_j)/C - 3Cm^i m_j/4 - l^i m_j/L,\\ &C_{\|j} = (b + C^2/4)m_j - Cl_j/L. \end{aligned}$$

which, because of (2. 30), yields

$$(5.\ 15)_2\qquad Lm^i|_j = -l^i m_j + L(C/4 + b/C)n^i n_j\ .$$

Comparing $(5.15)_2$ with (5. 12), we have

$$(5.16) \qquad d_j = -L(C/4 + b/C)n_j \ .$$

If $d_j = 0$, namely $b = -c^2/4$, then we differentiate this expression by y^h and substitute the result in (4. 12). Then we have

$$(5.17) \qquad C^2 L^2 = 8.$$

The v-curvature tensor is, because of (2. 26), given by

$$(5.18) \qquad L^2 S_{jikh} = S a_{ji} a_{kh} = S(h_{jk} h_{ih} - h_{jh} h_{ik}),$$

where $S = -C^2 L^2/8$. On the other hand, the indicatrix I_x at a point x of M is considered as a Riemannian space. In this case, the curvature tensor $\widetilde{S}_{jikh}$ of I_x is given by

$$(5.19) \qquad L^2 \widetilde{S}_{jikh} = L^2 S_{jikh} + (h_{jk} h_{ih} - h_{jh} h_{ik}).$$

Therefore if (5. 17) holds, then the tensor $\widetilde{S}_{jikh}$ vanishes because of (5. 18). However such a case should be excluded. In the following, we shall consider d_j as a non-zero vector. Consequently we can state

Theorem 15. *Let M be a three-dimensional C-reducible space with $C^2 P^2 \neq 0$. Then M is an IS-space if and only if M is a LCP-space such that relations (5. 2) and (5. 11)~(5. 13) hold, provided d_j is defined by (5. 16).*

We shall call a three-dimensional Finsler space M a *CR3IS*-space if M is a C-reducible *IS*-space with $C^2 P^2 \neq 0$.

Put $\lambda_j = m_i|_j n^i = m^i|_j n_i$. Then we have $-\lambda_j = n_i|_j m^i = n^i|_j m_i$.

Since $m^i|_j l_i = m^i|_j m_i = n^i|_j l_i = n^i|_j n_i 0$, tensors $m^i|_j$ and $n^i|_j$ are expressible in

$$(5.20) \qquad m^i|_j = n^i \lambda_j \ , \ n^i|_j = -m^i \lambda_j \ ,$$

which, because of (5. 1), implies

$$(5.21) \qquad a^i_k|_j = 0, \ T^i_k|_j = Lf|_j a^i_k \ .$$

From (5. 2) we have

$$(5.22) \qquad f|_i LC + fLC|_i = 3P|_i \ .$$

Since $P_i = C_{i|j}\, y^j$, $P_i = Pn_i$ and $C_i = Cm_i$, from (5. 20) and (5. 22) we have

(5. 23) $C_{|o} = 0,\ P = C\lambda_o,\ fL = 3\lambda_o,\ f_{|o}\, LC = 3P_{|o}$.

Let this space be of scalar curvature K with respect to the *IS*-connection. Then from Theorem 4 and (5. 1) we have

(5. 24) $K^i_{oh} = (K+f^2)L^2 h^i_h$.

Hence we can state

Theorem 16. *A CR3IS-space M is of scalar curvature K with respect to the IS-connection if and only if an equation (5. 24) holds. In this case, M is of scalar curvature* $K+f^2$ *in the usual sense.*

On making use of (1. 12), (2. 21), (5. 1), (5. 14) and (5. 21), the curvature tensor R^i_{kh} with respect to Γ^i_k is found as follows:

(5. 25) $R^i_{kh} = K^i_{kh} + L\,\{f_{|h}\, a^i_k - f_{|k}\, a^i_h + f^2(l_h h^i_k - l_k h^i_h) + fPn^i a_{kh}\}$.

Let M be n-flat. Then from (1. 18), (5. 1) and (5. 14) we first have

(5. 26) $K^i_{kh} = L\,\{f^2(l_k h^i_h - l_h h^i_k) + 4fPn^i a_{hk}/3\}$,

whitch implies $K_{i\,oh} = f^2 L^2 h_{ih}$. Next, from (5. 25) and (5. 26) we have

(5. 27) $f_{|h}\, a^i_k - f_{|k}\, a^i_h - fPn^i a_{hk}/3 = 0$.

Contracting (5. 27) by a^k_j, we have $-f_{|h}\, h^i_j - f_{|k}\, a^k_j a^i_h - fPn^i h_{jh}/3 = 0$, contraction of which with respect to i and j yields $-2f_{|h} + f_{|k}\, h^k_h - fPn_h/3 = 0$. Since $f_{|k}\, l^k = 0$, from the above expression we have

(5. 28) $f_{|h} = -\frac{1}{3} fPn_h$.

Consequently from Theorem 3 we can state

Theorem 17. *A CR3IS-space M is n-flat (or h-flat) with respect to the IS-connection if and only if equations (5. 26) and (5. 28) hold. In this case, M is of scalar curvature* f^2 *in the usual sense.*

Let M be of constant curvature K. Then the equation (1. 23) first holds. Now we shall calculate $\widetilde{T}^i_{j\,kh}$ in (1. 23). From (5. 2), (5. 4), (5. 5), (5. 11), (5. 12), (5. 13), (5. 15)$_1$ and (5. 16) we have

(5. 29) $P_{\| j} = \{-CPm_j + fL(b + C^2/4)m_j\} = (Lfb - CP/4)m_j$,

(5. 30) $$Ln^i{}_{\|j} = -\{l^i n_j + LCn^i m_j/4 + L(\tfrac{1}{2}C + b/C)m^i n_j\},$$

(5. 31) $$La_{hk\|j} = l_k a_{jk} + \tfrac{1}{2}LCa_{kh} m_j .$$

By the use of (5. 29) ~(5. 31), we have

(5. 32)
$$\begin{aligned}\widetilde{T}^i_{jkh} &= f^2(g_{jk}\delta^i_h - g_{jh}\delta^i_k) + 4fPn^i(l_k a_{jh} - l_h a_{jk})/3 \\ &\quad -2Pm_j(l_k h^i_h - l_h h^i_k) + 4a_{hk}\{fP(n^i l_j - l^i n_j) \\ &\quad -3p^2(\tfrac{1}{2} + b/C^2)m^i n_j + P^2(9b/C^2 - 4)n^i m_j\}/3.\end{aligned}$$

Next we shall calculate (1. 26). The tensor $\widetilde{T}^r_{jk}$ in (1. 26) is given by (5. 26). Therefore applying (5. 26) to (1. 26), we have

(5. 33)
$$\begin{aligned}&3f(f_{|j} l_k - f_{|k} l_j) + 2\{-(f_{|r}P + fP_{|r})n^r + fPm^r\lambda_r\} a_{jk} \\ &\quad + 2fP(n_j\lambda_k - n_k\lambda_j) + 2\{(f_{|k}P + fP_{|k})m_j - (f_{|j}P + fP_{|j})m_k\} = 0.\end{aligned}$$

If we contract (5. 33) by l^k and m^j respectively and use (5. 22) and (5. 23), then we obtain

(5. 34)
$$\begin{aligned}f_{|j} &= f_{|o} l_j/L - 4Cf_{|o} m_j/9 - 2Pfn_j/9, \\ P_{|k} &= Cf_{|o} l_k/3 + (P_{|r}m^r)m_k + (P_{|r}n^r)n_k, \\ \lambda_k &= fl_k/3 + (m^r\lambda_r)m_k + (n^r\lambda_r)n_k .\end{aligned}$$

Conversely if (5. 34) holds, then it is verified that (5. 33) holds.

Hence we can state

Theorem 18. *A CR3IS-space M is of constant curvature K with respect to the IS-connection if and only if equations (1. 23) and (5. 34) hold, provided that the tensor* $\widetilde{T}^i_{jkh}$ *in (1. 23) is given by (5. 32).*

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